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Noncommutative Ward's Conjecture and Integrable Systems

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Abstract

Noncommutative Ward's conjecture is a noncommutative version of the original Ward's conjecture which says that almost all integrable equations can be obtained from anti-self-dual Yang-Mills equations by reduction. In this paper, we prove that wide class of noncommutative integrable equations in both $(2+1)$- and $(1+1)$-dimensions are actually reductions of noncommutative anti-self-dual Yang-Mills equations with finite gauge groups, which include noncommutative versions of Calogero-Bogoyavlenskii-Schiff eq., Zakharov system, Ward's chiral and topological chiral models, (modified) Korteweg-de Vries, Non-Linear Schrödinger, Boussinesq, N-wave, (affine) Toda, sine-Gordon, Liouville, Tzitzéica, (Ward's) harmonic map eqs., and so on. This would guarantee existence of twistor description of them and the corresponding physical situations in N=2 string theory, and lead to fruitful applications to noncommutative integrable systems and string theories. Some integrable aspects of them are also discussed.

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1 Introduction

Non-commutative (NC) gauge theories are realized as effective theories of D-branes in background of magnetic fields and have been studied very intensively in string theories. (For reviews, see [1].) In some contexts, NC solitons are D-branes themselves and play key roles in the study of D-brane dynamics. In particular, NC anti-self-dual Yang-Mills (ASDYM) equations are very important because they are proved to be integrable in some sense and give rise to $U(1)$ instanton solutions special to NC theories [2]. (See also, [3])

NC extension of soliton theories and integrable systems is also one of hot topics and naively expected to be integrable as well and fruitful to yield new physical objects and to be applied to corresponding physical situations. (For reviews, see e.g. [4, 5].) However, these equations contain no gauge field and the equivalence between extension to NC spaces and presence of background magnetic fields is non-trivial. That is why it is very important to reveal the corresponding physical pictures of these NC integrable equations.[2]

On commutative spaces, there is a famous conjecture on integrable systems which was proposed by Richard Ward [6, 7] and often called *Ward's conjecture*. He conjectured in [6] that

many (and perhaps all?) of the ordinary and partial equations that are regarded as being integrable or solvable may be obtained from the anti-self-dual gauge field equations (or its generalizations) by reduction.

This is a very interesting conjecture which connects various lower-dimensional integrable equations in scalar theories with a master equation in gauge theories. Now we know that almost all famous integrable equations such as Korteweg-de-Vries (KdV) equation are actually derived from ASDYM equation in (2+2)-dimension by reduction and summarized in the book of Mason and Woodhouse [8]. (See also, e.g. [9, 10, 11].) This scenario can be embedded into geometrical framework of twistor theory and we can understand origins of integrability of lower dimensional integrable equations and classify them in some extent from the geometrical viewpoints.

ASDYM equations in $(2+2)$-dimension have the corresponding physical pictures in N=2 string theories [12] and NC extension of them are successful [13]. Hence if Ward's conjecture still holds on NC spaces, the reduced NC equations must have physical pictures

[2] In the present paper, integrability of NC equations means existence of infinite number of conserved quantities or exact multi-soliton solutions though the notion of integrability has not yet been established on NC $(1+1)$-dimensional space-time.

and various applications to the corresponding situations would be possible. Furthermore, NC twistor theories for NC ASDYM equation have been already developed by several authors in [14, 15, 16, 17, 18][3] for the Euclidean signature, (See also [19]), and in [20] for the ultrahyperbolic signature, which would leads to classification of NC integrable equations as in commutative case. That is why confirmation of NC Ward's conjecture is worth studying from the viewpoints of both N=2 string theories and twistor theories.

NC Ward's conjecture is first proposed in [21] as a future direction of integrable systems. Simple reductions of NC ASDYM equations in relation to the string theories have been already studied intensively, and exact soliton solutions are constructed with application to D-branes dynamics [19, 22, 23, 24, 25, 26, 27]. This suggests that all other NC integrable equations would also have physical situations and might lead to various successful applications to the corresponding D-brane dynamics and so on. Furthermore, several nontrivial reductions of NC ASDYM equations have been successful for NC NLS eq. [28], NC supersymmetric KdV eq. [28, 29], NC Burgers eq. [30], NC sine-Gordon eq. [31, 32], NC (affine) Toda and NC Liouville equations [33], NC KdV, N-wave, Kadomtsev-Petviashvili (KP), Davey-Stewartson (DS) equations [34]. All of the reduced equations have integrable-like properties, such as, linealizability, existence of infinite conserved quantities or exact multi-soliton solutions and so on, which might be explained clearly in the geometric framework of NC twistor theories.

These results on NC Ward's conjecture have been examined individually by focusing on each equation and arguments with *symmetry* in the reduction are missing. Furthermore, most of the reduced equations belong to $(1+1)$-dimension,[4] in which integrability is hard to discuss and even to define because noncommutativity must be introduced into time direction.

In this paper, we show that NC integrable equations in both $(2+1)$- and $(1+1)$-dimensions are actually reductions of noncommutative anti-self-dual Yang-Mills equations with *finite* gauge groups, which include new non-trivial reductions to Calogero-Bogoyavlenskii-Schiff (CBS) eq. and Zakharov system in $(2+1)$-dimension and mKdV, Boussinesq, Tzitzéica and harmonic map eqs. and topological chiral model in $(1+1)$-dimension. The discussion is clarified from the viewpoint of symmetry reduction in some extent. This would lead to a systematic classification of NC integrable systems in the

[3]The author thanks Brain and Hannabuss for informing him of their new results [17, 18].

[4]In [34], several NC equations in $(2+1)$-dimension, such as KP and DS equations, are derived from NC ASDYM equation with *infinite* gauge groups. These reductions have no naive twistor description on commutative spaces and would not be considered to be examples of Ward's conjecture.

framework of NC twistor theory and to fruitful applications to the corresponding situations in N=2 string theory. Now we can say that NC Ward's conjecture has been almost confirmed by the explicit examples. We also show gauge equivalence of reductions to potential KdV, KdV and modified KdV (mKdV) eqs. and develop NC Bäcklund transformation for NC ASDYM equation in some extent, which would be applied to lower-dimensional equations by reduction.

This paper is organized as follows. In section 2, we discuss some aspects of of NC ASDYM equations including Yang's forms and Bäcklund transformations and summarize the strategy of reductions of the NC ASDYM equations. In section 3, we perform reductions of NC ASDYM to NC integrable equations $(2+1)$-dimension. In section 4, we give various examples of NC Ward's conjecture into $(1+1)$-dimension. We comment on the integrable properties of the reduced equations at each example of the reduction. In section 5, we conclude and discuss further directions.

2 NC ASDYM equation

In this section, we review some aspects of NC ASDYM equation and establish notations. Discussion on NC Bäcklund transformation is new.

2.1 NC gauge theory

NC spaces are defined by the noncommutativity of the coordinates:

$$[x^i, x^j] = i\theta^{ij}, \tag{2.1}$$

where θ^{ij} are real constants called the *NC parameters*. The NC parameter is anti-symmetric with respect to i,j: $\theta^{ji} = -\theta^{ij}$ and the rank is even. This relation looks like the canonical commutation relation in quantum mechanics and leads to "space-space uncertainty relation." Hence the singularity which exists on commutative spaces could resolve on NC spaces. This is one of the prominent features of NC field theories and yields various new physical objects.

NC field theories are given by the exchange of ordinary products in the commutative field theories for the star-products and realized as deformed theories from the commutative ones. The ordering of non-linear terms are determined by some additional requirements such as gauge symmetry. The star-product is defined for ordinary fields on commutative

spaces. For Euclidean spaces, it is explicitly given by

$$\begin{aligned} f \star g(x) &:= \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + O(\theta^2), \end{aligned} \tag{2.2}$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. This explicit representation is known as the *Moyal product* [35]. The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$, and returns back to the ordinary product in the commutative limit: $\theta^{ij} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative," that is, $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual, for example, $\partial_i \star \partial_j = \partial_i \partial_j$, and the wedge product of $\omega = \omega_i(x)dx^i$ and $\lambda = \lambda_j(x)dx^j$ is $\omega_i \star \lambda_j dx^i \wedge dx^j$.

NC gauge theories are defined in this way by imposing NC version of gauge symmetry, where the gauge transformation is defined as follows:

$$A_\mu \to g^{-1} \star A_\mu \star g + g^{-1} \star \partial_\mu g, \tag{2.3}$$

where g is an element of the gauge group G. This is sometimes called the *star gauge transformation* [1]. We note that because of noncommutativity, the commutator terms in field strength are always needed even when the gauge group is abelian in order to preserve the star gauge symmetry. This $U(1)$ part of the gauge group actually plays crucial roles in general. We note that because of noncommutativity of matrix elements, cyclic symmetry of traces is broken in general:

$$\mathrm{Tr}\ A \star B \neq \mathrm{Tr}\ B \star A. \tag{2.4}$$

Therefore, gauge invariant quantities becomes hard to define on NC spaces.

2.2 NC ASDYM equation

Let us consider Yang-Mills theories on $(2+2)$-dimensional NC spaces whose real coordinates of the space are denoted by (x^0, x^1, x^2, x^3), where the gauge group is $GL(N, \mathbb{C})$. Here, we follow the convention in [8] as follows.

First, we introduce double null coordinates of $(2+2)$-dimensional space as follows

$$ds^2 = 2(dz d\tilde{z} - dw d\tilde{w}), \tag{2.5}$$

where

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}. \tag{2.6}$$

The reality conditions are $\bar{z} = \tilde{z}, \bar{w} = \tilde{w}$.[5] The coordinate vectors $\partial_z, \partial_z.\partial_{\tilde{w}}, \partial_{\tilde{z}}$ form a null tetrad and are represented explicitly as:

$$\begin{aligned} \partial_z &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^0} + i \frac{\partial}{\partial x^1} \right), \quad \partial_{\tilde{z}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^0} - i \frac{\partial}{\partial x^1} \right), \\ \partial_w &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^2} + i \frac{\partial}{\partial x^3} \right), \quad \partial_{\tilde{w}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x^2} - i \frac{\partial}{\partial x^3} \right). \end{aligned} \tag{2.7}$$

For ultrahyperbolic signature, Hodge dual operator $*$ satisfies $*^2 = 1$ and hence the space of 2-forms β decomposes into the direct sum of eigenvalues of $*$ with eigenvalues ± 1, that is, self-dual (SD) part ($*\beta = \beta$) and anti-self-dual (ASD) part ($*\beta = -\beta$).

Typical examples of SD forms are

$$\alpha = dw \wedge dz, \quad \tilde{\alpha} = d\tilde{w} \wedge d\tilde{z}, \quad \omega = dw \wedge d\tilde{w} - dz \wedge d\tilde{z}, \tag{2.8}$$

and those of ASD forms are

$$dw \wedge d\tilde{z}, \quad d\tilde{w} \wedge dz, \quad dw \wedge d\tilde{w} + dz \wedge d\tilde{z}. \tag{2.9}$$

NC ASDYM equation is derived from compatibility condition of the following linear system:

$$\begin{aligned} L \star \Psi &:= (D_w - \lambda D_{\tilde{z}}) \star \Psi = (\partial_w + A_w - \lambda(\partial_{\tilde{z}} + A_{\tilde{z}})) \star \Psi(x; \lambda) = 0, \\ M \star \Psi &:= (D_z - \lambda D_{\tilde{w}}) \star \Psi = (\partial_z + A_z - \lambda(\partial_{\tilde{w}} + A_{\tilde{w}})) \star \Psi(x; \lambda) = 0, \end{aligned} \tag{2.10}$$

where $A_z, A_w, A_{\tilde{z}}, A_{\tilde{w}}$ and $D_z, D_w, D_{\tilde{z}}, D_{\tilde{w}}$ denote gauge fields and covariant derivatives in the Yang-Mills theory, respectively. The constant λ is called the *spectral parameter*. The compatible condition $[L, M]_\star = 0$ gives rise to a quadratic polynomial of λ and each coefficient yields NC ASDYM equations whose explicit representations are as follows:

$$\begin{aligned} F_{wz} &= \partial_w A_z - \partial_z A_w + [A_w, A_z]_\star = 0, \\ F_{\tilde{w}\tilde{z}} &= \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}]_\star = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} &= \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_z, A_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star = 0. \end{aligned} \tag{2.11}$$

[5]For Euclidean signature, we have only to put another reality condition as $\bar{z} = \tilde{z}, \bar{w} = -\tilde{w}$.

As in commutative case, if Π is a null 2-plane in space-time, a tangent bivector π associated to Π satisfies $\pi_{\mu\nu}\pi^{\mu\nu} = 0$ and $\pi_{\mu\nu}dx^\mu \wedge dx^\nu$ is either SD or ASD. For example, vector fields $l = \partial_w - \lambda\partial_{\tilde{z}}$ and $m = \partial_z - \lambda\partial_{\tilde{w}}$ form a SD bivector $l \wedge m$. A 2-plane Π is called an *α-plane* when the associated tangent bivector π is SD. NC ASDYM eqs are equivalent to the condition that the connection is flat on the α-plane: $F(l, m) = 0$. (For detailed discussion, see e.g. [8].)

2.3 NC Yang's equation and J, K-matrices

Here we discuss the potential forms of NC ASDYM equations such as NC J, K-matrix formalisms and NC Yang's equation, which is already presented by e.g. K. Takasaki [15]. (See also [36].)

Let us first discuss the *J-matrix formalism* of NC ASDYM equation. The first equation of NC ASDYM equation (2.11) is the compatible condition of $D_z \star h = 0, D_w \star h = 0$, where h is a $N \times N$ matrix. Hence we get

$$A_z = -h_z \star h^{-1}, \quad A_w = -h_w \star h^{-1}, \tag{2.12}$$

where $h_z := \partial h/\partial z,\ h_w := \partial h/\partial w$. Similarly, the second eq. of NC ASDYM equation (2.11) leads to

$$A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \star \tilde{h}^{-1}, \quad A_{\tilde{w}} = -\tilde{h}_{\tilde{w}} \star \tilde{h}^{-1}, \tag{2.13}$$

where $\tilde{h}$ is also a $N \times N$ matrix. By defining $J = \tilde{h}^{-1} \star h$, the third eq. of NC ASDYM equation (2.11) becomes NC Yang's equation

$$\partial_z(J^{-1} \star \partial_{\tilde{z}} J) - \partial_w(J^{-1} \star \partial_{\tilde{w}} J) = 0. \tag{2.14}$$

This representation is useful for reductions to NC Ward's chiral models and NC sine-Gordon, Liouville and Tzitzéica equations, and so on.

There is another potential form of NC ASDYM equation, known as the *K-matrix formalism*. Under the gauge $A_w = A_z = 0$, the third eq. of NC ASDYM equation (2.11) becomes $\partial_z A_{\tilde{z}} - \partial_w A_{\tilde{w}} = 0$, which implies existence of a potential K such that $A_{\tilde{z}} = \partial_w K, A_{\tilde{w}} = \partial_z K$. Then the second eq. of NC ASDYM equation (2.11) becomes

$$\partial_z\partial_{\tilde{z}} K - \partial_w\partial_{\tilde{w}} K + [\partial_w K, \partial_z K]_\star = 0. \tag{2.15}$$

2.4 NC Bäcklund transformation

Here we make a brief discussion on hidden symmetry behind Yang's equation and NC Bäcklund transformation for NC ASDYM equation. This would relates to NC Bäcklund transformation for NC integrable equations in lower dimensions via reductions.

First let us extend the discussion on hidden symmetry for Yang's equation [8] to NC spaces. A generic $N \times N$ matrix J can be decomposed as follows:

$$J = \begin{pmatrix} A^{-1} - \tilde{B} \star \tilde{A} \star B & -\tilde{B} \star \tilde{A} \\ \tilde{A} \star B & \tilde{A} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{B} \\ 0 & \tilde{A}^{-1} \end{pmatrix}^{-1} \star \begin{pmatrix} A^{-1} & 0 \\ B & 1 \end{pmatrix}, \qquad (2.16)$$

where $A, \tilde{A}, B$ and $\tilde{B}$ are $k \times k$, $\tilde{k} \times \tilde{k}$, $\tilde{k} \times k$ and $k \times \tilde{k}$ matrices, respectively, with $k + \tilde{k} = N$ and A and $\tilde{A}$ are non-singular. With this decomposition, NC Yang's equation (2.14) becomes

$$\begin{aligned} &\partial_z(A \star \tilde{B}_{\tilde{z}} \star \tilde{A}) - \partial_w(A \star \tilde{B}_{\tilde{w}} \star \tilde{A}) = 0, \quad \partial_{\tilde{z}}(\tilde{A} \star B_z \star A) - \partial_{\tilde{w}}(\tilde{A} \star B_w \star A) = 0, \\ &\partial_z(\tilde{A}^{-1} \star \tilde{A}_{\tilde{z}}) \star \tilde{A}^{-1} - \partial_w(\tilde{A}^{-1} \star \tilde{A}_{\tilde{w}}) \star \tilde{A}^{-1} + B_z \star A \star \tilde{B}_{\tilde{z}} - B_w \star A \star \tilde{B}_{\tilde{w}} = 0, \\ &A^{-1} \star \partial_z(A_{\tilde{z}} \star A^{-1}) - A^{-1} \star \partial_w(A_{\tilde{w}} \star \tilde{A}^{-1}) + \tilde{B}_{\tilde{z}} \star \tilde{A} \star B_z - \tilde{B}_{\tilde{w}} \star \tilde{A} \star B_w = 0. \end{aligned} \qquad (2.17)$$

The first two equations can be interpreted as integrability conditions. Hence there exist $k \times \tilde{k}$ and $\tilde{k} \times k$ matrices B^{new} and $\tilde{B}^{\text{new}}$, respectively, such that

$$\begin{aligned} \partial_z B^{\text{new}} &= A \star \tilde{B}_{\tilde{w}} \star \tilde{A}, \quad \partial_w B^{\text{new}} = A \star \tilde{B}_{\tilde{z}} \star \tilde{A}, \\ \partial_{\tilde{z}} \tilde{B}^{\text{new}} &= \tilde{A} \star B_w \star A, \quad \partial_{\tilde{w}} \tilde{B}^{\text{new}} = \tilde{A} \star B_z \star A. \end{aligned} \qquad (2.18)$$

Furthermore, if we put

$$\begin{aligned} &A^{\text{new}} = \tilde{A}^{-1}, \ \tilde{A}^{\text{new}} = A^{-1}, \\ &k^{\text{new}} = \tilde{k}, \ \tilde{k}^{\text{new}} = k, \end{aligned} \qquad (2.19)$$

then we obtain a new solution J^{new} of NC Yang's equation. This can be considered as a transformation $i_k : J \to J^{\text{new}}$. We can see that $i_{n-k} \circ i_k$ is identity.

This can be understood from the viewpoint of NC Bäcklund transformation for NC ASDYM equation in the following way. If we take

$$h = \begin{pmatrix} A^{-1} & 0 \\ B & 1 \end{pmatrix}, \quad \tilde{h} = \begin{pmatrix} 1 & \tilde{B} \\ 0 & \tilde{A}^{-1} \end{pmatrix}, \qquad (2.20)$$

then we can choose the gauge so that

$$A_z = -h_z \star h^{-1}, \quad A_w = -h_w \star h^{-1}, \quad A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \star \tilde{h}^{-1}, \quad A_{\tilde{w}} = -\tilde{h}_{\tilde{w}} \star \tilde{h}^{-1}, \qquad (2.21)$$

and the linear systems of NC ASDYM is

$$\begin{aligned} L &= \partial_w - \lambda\partial_{\tilde{z}} + \begin{pmatrix} A^{-1}\star A_w & \lambda\tilde{B}_{\tilde{z}}\star\tilde{A} \\ -B_w\star A & -\lambda\tilde{A}^{-1}\star\tilde{A}_{\tilde{z}} \end{pmatrix}, \\ M &= \partial_z - \lambda\partial_{\tilde{w}} + \begin{pmatrix} A^{-1}\star A_z & \lambda\tilde{B}_{\tilde{w}}\star\tilde{A} \\ -B_z\star A & -\lambda\tilde{A}^{-1}\star\tilde{A}_{\tilde{w}} \end{pmatrix}. \end{aligned} \tag{2.22}$$

Now let us consider the following gauge transformation

$$L^{\text{new}} = g^{-1}\star L\star g, \quad M^{\text{new}} = g^{-1}\star M\star g, \tag{2.23}$$

where

$$g^{-1} = \begin{pmatrix} 0 & \lambda\tilde{A} \\ -A & 0 \end{pmatrix}. \tag{2.24}$$

The transformed linear systems are as follows

$$\begin{aligned} L^{\text{new}} &= \partial_w - \lambda\partial_{\tilde{z}} + \begin{pmatrix} -\tilde{A}_{\tilde{z}}\star\tilde{A}^{-1} & \lambda\tilde{A}\star B_w \\ -A\star\tilde{B}_{\tilde{z}} & \lambda A_{\tilde{z}}\star A^{-1} \end{pmatrix}, \\ M^{\text{new}} &= \partial_z - \lambda\partial_{\tilde{w}} + \begin{pmatrix} -\tilde{A}_{\tilde{w}}\star\tilde{A}^{-1} & \lambda\tilde{A}\star B_z \\ -A\star\tilde{B}_{\tilde{w}} & \lambda A_{\tilde{w}}\star A^{-1} \end{pmatrix}. \end{aligned} \tag{2.25}$$

Now by making the identification just as (2.18) and (2.19), L^{new} and M^{new} have the same form as Eq. (2.22). Hence the hidden symmetry can be interpreted as a Bäcklund transformation for the linear systems of NC ASDYM equation.

We note that the gauge transformation (2.23) is irregular in the sense that the determinant of g becomes singular at $\lambda = 0, \infty$. However, we can treat this point properly as in commutative case because the spectral parameter λ is a commutative variable. (See also [15].) More detailed discussion will be reported in a separated paper. (For commutative discussion, see e.g. [37, 8]. Bäcklund transformation for a few NC integrable equations is briefly discussed in e.g. [38].)

2.5 Reduction of NC ASDYM equation

Here we summarize the strategy for reductions of NC ASDYM equation into lower-dimensions. Reductions are classified by the following ingredients:

- A choice of gauge group
- A choice of symmetry, such as, translational symmetry

- A choice of gauge fixing
- A choice of constants of integrations in the process of reductions

Gauge groups are in general $GL(N,\mathbb{C})$. We have to take $U(1)$ part into account in NC case. A choice of symmetry reduces NC ASDYM equations to simple forms. We note that noncommutativity must be eliminated in the reduced directions because of compatibility with the symmetry. Hence within the reduced directions, discussion about the symmetry is the same as commutative one. A choice of gauge fixing is the most important ingredient in this paper which is shown explicitly at each subsection. The residual gauge symmetry sometimes shows equivalence of a few reductions as we will see in Secs. 4.1 and 4.2. Constants of integrations in the process of reductions sometimes lead to parameter families of NC reduced equations, however, in this paper, we set all integral constants zero for simplicity.

In the following sections, we present explicit reductions of NC ASDYM equation to various NC integrable equation classified in the above viewpoints.

Before discussing the reductions, we comment on how to guess the reduction of NC ASDYM equation to the relevant lower-dimensional NC integrable equations. Because the original ASDYM equation possesses the Lax representation in a wider sense, the reduced equations always possess the Lax representations. The Lax representations are common in many integrable equations. That is one of the reasons why Ward's conjecture is reasonable. Oppositely, in some case, Lax representations of integrable systems with a spectral parameter in $(1+1)$-dimensions can be embedded into Lax form of ASDYM equation. In order to see this, for example, let us consider a Lax formalism of NC KdV equation:

$$P\star\psi=(\partial_x^2+u)\star\psi=\lambda\psi,\quad B\star\psi=(\partial_t-\partial_x^3-\frac{3}{2}u\partial_x-\frac{3}{4}u')\star\psi=0, \tag{2.26}$$

where $u':=\partial u/\partial x, \dot{u}:=\partial u/\partial t$. The compatibility condition $[P,B]_\star=0$ gives rise to NC KdV equation $4\dot{u}-u'''+3u'\star u+3u\star u'=0$.

Now let us introduce

$$\Psi=\left(\begin{array}{c}\psi\\ \psi_x\end{array}\right), \tag{2.27}$$

then the derivatives of Ψ with respect to x and t are calculated from Eq. (2.26) as follows

$$\partial_x\Psi \;=\; \left\{\left(\begin{array}{cc}0&1\\-u&0\end{array}\right)+\lambda\left(\begin{array}{cc}0&0\\1&0\end{array}\right)\right\}\left(\begin{array}{c}\psi\\ \psi_x\end{array}\right), \tag{2.28}$$

$$
\begin{aligned}
\partial_t \Psi &= \left\{ \frac{1}{4} \begin{pmatrix} -u' & 2u \\ -u'' - 2u \star u & u' \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ -(1/2)u & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \\
&= \left\{ \frac{1}{4} \begin{pmatrix} -u' & 2u \\ -u'' - 2u \star u & u' \end{pmatrix} + \lambda \left[\partial_x + \begin{pmatrix} 0 & 0 \\ (1/2)u & 0 \end{pmatrix} \right] \right\} \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}.
\end{aligned}
\tag{2.29}
$$

We note that it is nontrivial that Eq. (2.29) becomes linear with respect to λ as in the second line of Eq. (2.29). From this representation, we can find a candidate for a reduction to NC KdV equation of NC ASDYM equation by identifying the linear systems (2.10) with Eqs. (2.29) and (2.29):

$$
A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ u/2 & 0 \end{pmatrix}, A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}, A_z = \frac{1}{4} \begin{pmatrix} u' & -2u \\ u'' + 2u \star u & -u' \end{pmatrix},
$$

together with $\partial_w = \partial_{\tilde{w}} = \partial_x$ and $\partial_{\tilde{z}} = 0$. This actually coincides with the reduction condition to NC KdV equation from NC ASDYM equation as in Eq. (4.8).

3 Reduction to $(2+1)$-dimension

In this section, we present non-trivial reductions of NC ASDYM equation to $(2+1)$-dimension. Reductions to NC CBS equation and NC Zakharov system are new results.

First, let us take a reduction by a null translation:

$$
Y = \partial_{\tilde{z}}. \tag{3.1}
$$

The gauge field $A_{\tilde{z}}$ becomes a Higgs field which is denoted by $\Phi_{\tilde{z}}$.

For $G = GL(2, \mathbb{C})$, standard choices of gauge fixing of $\Phi_{\tilde{z}}$ are as follows:

$$
\text{(i)} \quad \Phi_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{(ii)} \quad \Phi_{\tilde{z}} = \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

where κ is a constant. By taking a gauge $A_{\tilde{w}} = 0$, we get a reduced NC ASDYM equation:

$$
\partial_{\tilde{w}} A_w + [A_z, \Phi_{\tilde{z}}]_\star = 0, \tag{3.2}
$$

$$
\partial_w A_z - \partial_z A_w + [A_w, A_z]_\star = 0. \tag{3.3}
$$

In the following, we will see the cases (i) and (ii) give rise to NC CBS equation and NC Zakharov system which are $(2+1)$-dimensional generalizations of NC KdV equation and NC NLS equation, respectively.

3.1 Reduction to NC CBS equation

The Calogero-Bogoyavlenskii-Schiff (CBS) equation[6] is a $(2+1)$-dimensional generalization of the KdV equation and one of the important equations in $(2+1)$-dimension. NC version of it was first derived in [42]. Here, we derive the equation from NC ASDYM equation by reduction as follows. For the commutative discussion, see [8, 41].

Now let us take further reduction on the gauge fields in the ASDYM equation (3.2) and (3.3) as follows:

$$A_w = \begin{pmatrix} q & -1 \\ q_w + q \star q & -q \end{pmatrix}, \ A_z = \begin{pmatrix} (1/2)q_{w\tilde{w}} + q_{\tilde{w}} \star q + \alpha & -q_{\tilde{w}} \\ \phi & -(1/2)q_{w\tilde{w}} - q \star q_{\tilde{w}} + \alpha \end{pmatrix},$$

where α and ϕ are differential polynomials of q. The gauge group is $GL(2,\mathbb{R})$. The ordering of nonlinear terms in the trace part of A_z is determined in order that in the case of $\partial_w = \partial_{\tilde{w}}$, this reduction to NC CBS equation should coincide with that to NC KdV equation. The unknown variable α also satisfies the condition that α should vanish in the both cases of the commutative limit and $\partial_w = \partial_{\tilde{w}}$.

Eq. (3.2) is trivially satisfied. Eq. (3.3) yields

$$\phi = -q_z + \frac{1}{2}q_{ww\tilde{w}} + \frac{1}{2}\{q, q_{w\tilde{w}}\}_\star + q \star q_{\tilde{w}} \star q + 2q_{\tilde{w}} \star q_w + \alpha_w + [q, \alpha]_\star, \qquad (3.4)$$

$$\phi = -q_z + \frac{1}{2}q_{ww\tilde{w}} + \frac{1}{2}\{q, q_{w\tilde{w}}\}_\star + q \star q_{\tilde{w}} \star q + 2q_w \star q_{\tilde{w}} - \alpha_w + [q, \alpha]_\star, \qquad (3.5)$$

$$\begin{aligned}&(\phi - q_z)_w - \{q, \phi + q_z\}_\star + [q_w + q \star q, \alpha]_\star + q_w \star q_{\tilde{w}} \star q + q \star q_{\tilde{w}} \star q_w \\ &\quad + \frac{1}{2}\{q_w, q_{w\tilde{w}}\}_\star + \frac{1}{2}\{q \star q, q_{w\tilde{w}}\}_\star + \{q, q \star q_{\tilde{w}} \star q\}_\star = 0. \qquad (3.6)\end{aligned}$$

where $\{A, B\}_\star := A \star B + B \star A$.

From Eqs. (3.4) and (3.5), we get $\alpha = \partial_w^{-1}[q_w, q_{\tilde{w}}]_\star$, where $\partial_w^{-1} f(w) := \int^w dw' f(w')$. This satisfies the condition for α.

Then we get

$$\phi = -q_z + \frac{1}{2}q_{ww\tilde{w}} + \frac{1}{2}\{q, q_{w\tilde{w}}\}_\star + \frac{1}{2}\{q_w, q_{\tilde{w}}\}_\star + q \star q_{\tilde{w}} \star q + [q, \partial_w^{-1}[q_w, q_{\tilde{w}}]_\star]_\star, \qquad (3.7)$$

and Eq. (3.6) becomes NC potential CBS equation:

$$q_{zw} = \frac{1}{4}q_{www\tilde{w}} + \{q_w, q_{w\tilde{w}}\}_\star + \frac{1}{2}\{q_{\tilde{w}}, q_{ww}\}_\star + [q_w, \partial_w^{-1}[q_w, q_{\tilde{w}}]_\star]_\star, \qquad (3.8)$$

[6] As far as the author knows, this equation was obtained first by Calogero [39], and Bogoyavlenskii [40], and Schiff [41] from different viewpoints, probably independently. That is why we use the present name.

which is derived from the NC CBS equation [42, 21]

$$u_z = \frac{1}{4}u_{ww\tilde{w}} + \frac{1}{2}\{u, u_{\tilde{w}}\}_\star + \frac{1}{4}\left\{u_{\tilde{w}}, \partial_w^{-1}u_{\tilde{w}}\right\}_\star + \frac{1}{4}\partial_w^{-1}[u, \partial_w^{-1}[u, \partial_w^{-1}u_{\tilde{w}}]_\star]_\star \tag{3.9}$$

by setting $2q_w = u$.

If we impose an additional symmetry described by non-null translation $X = \partial_w - \partial_{\tilde{w}}$, which is realized by the identification $\partial_w = \partial_{\tilde{w}}$, the present discussion is reduced to that to NC KdV equation [34], which is NC generalization of Mason-Sparling's result [43]. In this sense, NC CBS equation is a $(2+1)$-dimensional extension of NC KdV equation [42, 21].

3.2 Reduction to NC Zakharov system

Zakharov system [44] is a $(2+1)$-dimensional extension of NLS equation in the same sense as CBS equation. Here we take the choice (ii) for $\Phi_{\tilde{z}}$, and make the same procedure to yield NC Zakharov system from NC ASDYM equation as NC CBS equation. For the commutative discussion, see [8, 45].

Here let us take another further reduction on the gauge fields in the ASDYM equation (3.2) and (3.3) as follows:

$$A_w = \begin{pmatrix} 0 & \psi \\ \tilde{\psi} & 0 \end{pmatrix},\ A_z = \frac{1}{2\kappa}\begin{pmatrix} U & \psi_{\tilde{w}} \\ -\tilde{\psi}_{\tilde{w}} & V \end{pmatrix}, \tag{3.10}$$

where U and V are differential polynomials of ψ and $\tilde{\psi}$. The off-diagonal elements in A_z are determined in order that Eq. (3.2) should be solved. Each component of Eq. (3.3) becomes

$$\begin{aligned} &U_w - \psi\star\tilde{\psi}_{\tilde{w}} - \psi_{\tilde{w}}\star\tilde{\psi} = 0,\ V_w + \tilde{\psi}\star\psi_{\tilde{w}} + \tilde{\psi}_{\tilde{w}}\star\psi = 0, \\ &\psi_{w\tilde{w}} - 2\kappa\psi_z + \psi\star V - U\star\psi = 0,\ \ -\tilde{\psi}_{w\tilde{w}} - 2\kappa\psi_z + \tilde{\psi}\star U - V\star\tilde{\psi} = 0. \end{aligned}$$

The first two equations lead to $U = \partial_w^{-1}\partial_{\tilde{w}}(\psi\star\tilde{\psi}),\ V = -\partial_w^{-1}\partial_{\tilde{w}}(\tilde{\psi}\star\psi)$. By taking $\kappa = i/2, \tilde{\psi} = \varepsilon\bar{\psi}$ where $\varepsilon = \pm 1$, the remaining two equations coincide with NC Zakharov system:

$$i\psi_z \ \ = \ \ \psi_{w\tilde{w}} - \varepsilon\psi\star\partial_w^{-1}\partial_{\bar{w}}(\bar{\psi}\star\psi) - \varepsilon\partial_w^{-1}\partial_{\bar{w}}(\psi\star\bar{\psi})\star\psi, \tag{3.11}$$

and the complex conjugate. Now the gauge group reduces to $U(1,1)$ or $U(2)$. The present discussion reduces to that to NC NLS equation by identifying $\partial_w = \partial_{\tilde{w}}$. This system is

studied in more general framework by Dimakis and Müller-Hoissen and proved to possess infinite conserved quantities [46] in terms of Strachan's product [47].

Zakharov system can be embedded into framework of a generalized twistor theory [48]. The NC extension is worth studying.

3.3 Reduction to NC Ward's chiral model

Here let us start with NC Yang's Eq. (2.14) and take a simple dimensional reduction by non-null transformation such as $X = \partial_w - \partial_{\tilde{w}}$. The reduced equation is NC Ward's chiral model:

$$(\eta^{ij} + \epsilon^{ij2})\partial_i(J^{-1} \star \partial_j J) = 0, \tag{3.12}$$

where η^{ij} $(i,j = 0,1,2)$ is the Minkowski metric: $\eta^{ij} = \mathrm{diag}(+,+,-)$, and ϵ^{ijk} is a totally anti-symmetric tensor with $\epsilon_{012} = 1$. This equation has been studied intensively and proved to be integrable in the sense that dressing and splitting methods can work well [22]-[27].

4 Reduction to $(1+1)$-dimension

In this section, we present various reductions of NC ASDYM into $(1+1)$-dimensions. In this case, two kind of translational invariance are imposed, which is classified here as follows:

- H_{+0}: generated by $X = \partial_w - \partial_{\tilde{w}}$, $Y = \partial_{\tilde{z}}$, which includes KdV, mKdV and NLS equations. The reduced NC ASDYM equation is

$$\begin{aligned} &\Phi'_{\tilde{z}} + [A_{\tilde{w}}, \Phi_{\tilde{z}}]_\star = 0, \\ &\dot{\Phi}_{\tilde{z}} + A'_w - A'_{\tilde{w}} + [A_z, \Phi_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star = 0, \\ &A'_z - \dot{A}_w + [A_w, A_z]_\star = 0, \end{aligned} \tag{4.1}$$

  where $(t,x) \equiv (z, w + \tilde{w})$ and $\dot{f} := \partial f/\partial t$, $f' := \partial f/\partial x$.[7]

- H_{SD}: generated by $X = \partial_{\tilde{w}}$, $Y = \partial_{\tilde{z}}$, which includes Boussinesq and N-wave equations, and topological chiral model. The reduced NC ASDYM equation is

$$\begin{aligned} &[\Phi_{\tilde{w}}, \Phi_{\tilde{z}}]_\star = 0, \;\; A'_z - \dot{A}_w + [A_w, A_z]_\star = 0, \\ &\dot{\Phi}_{\tilde{z}} - \Phi'_{\tilde{w}} + [A_z, \Phi_{\tilde{z}}]_\star - [A_w, \Phi_{\tilde{w}}]_\star = 0, \end{aligned} \tag{4.2}$$

[7] Note that in our convention, $\partial_x = \partial_w = \partial_{\tilde{w}}$. (See, Eq. (2.7).)

where $(t, x) \equiv (z, w)$.

- H_{++}: generated by $X = \partial_w,\ Y = \partial_{\tilde{w}}$, which includes (affine) Toda, sine-Gordon, Liouville, Tzitzéica, and Ward's harmonic map and chiral equations and so on. The reduced NC ASDYM equation is

$$\begin{aligned}&\partial_z \Phi_w + [A_z, \Phi_w]_\star = 0,\ \partial_{\tilde{z}} \Phi_{\tilde{w}} + [A_{\tilde{z}}, \Phi_{\tilde{w}}]_\star = 0,\\&\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + [A_z, A_{\tilde{z}}]_\star + [\Phi_{\tilde{w}}, \Phi_w]_\star = 0,\end{aligned} \tag{4.3}$$

where we follow the notations in [8] as well. Only in Sec. 4.11, we choose a different choice of X, Y which is defined in the relevant part.

4.1 Reduction to NC KdV equation

As we commented in Sec. 3.1, by the identification $\partial_w = \partial_{\tilde{w}}$, the reduction to NC CBS equation coincides with that [34] to NC KdV equation, that is, the following reduction conditions

$$\begin{aligned}&A_{\tilde{w}} = 0,\ \Phi_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_w = \begin{pmatrix} q & -1 \\ q' + q \star q & -q \end{pmatrix},\\&A_z = \frac{1}{2}\begin{pmatrix} q'' + 2q' \star q & -2q' \\ (1/2)q''' + q' \star q' + \{q, q''\}_\star + 2q \star q' \star q & -q'' - 2q \star q' \end{pmatrix},\end{aligned} \tag{4.4}$$

gives rise to NC potential KdV equation

$$\dot{q} = \frac{1}{4} q''' + \frac{3}{2} (q \star q)', \tag{4.5}$$

which is derived from NC KdV equation with $u = 2q'$

$$\dot{u} = \frac{1}{4} u''' + \frac{3}{4} (u' \star u + u \star u'). \tag{4.6}$$

The present reduction belongs to H_{+0} where the gauge group is $GL(2, \mathbb{R})$. NC KdV equation has been studied by several authors and proved to possess infinite conserved quantities [49] (See also [50].) and exact multi-soliton solutions [51, 52].

Now we comment on a gauge equivalent reduction. First we note that the following gauge transformation for (4.4) leaves $\Phi_{\tilde{z}}$ as it is:

$$A_\mu \to A_\mu^{(i)} = g_{(i)}^{-1} \star A_\mu \star g_{(i)} + g_{(i)}^{-1} \star \partial_\mu g_{(i)}, \quad g_{(i)} = \begin{pmatrix} 1 & 0 \\ h_{(i)} & 1 \end{pmatrix}. \tag{4.7}$$

If we take $h_{(1)} = q$,[8] the transformed gauge fields are calculated as

$$\begin{aligned}
&A_{\tilde{w}}^{(1)} = \begin{pmatrix} 0 & 0 \\ u/2 & 0 \end{pmatrix}, \ \Phi_{\tilde{z}}^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ A_w^{(1)} = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}, \\
&A_z^{(1)} = \frac{1}{4}\begin{pmatrix} u' & -2u \\ u'' + 2u \star u & -u' \end{pmatrix},
\end{aligned} \tag{4.8}$$

with $u = 2q'$. This is a gauge equivalent, direct reduction to NC KdV equation. This coincides with the discussion in Sec. 4.1 in [34] by identifying $(\Phi_{\tilde{z}}^{(1)} A_{\tilde{w}}^{(1)}, A_z^{(1)}, A_w^{(1)})$ with $(\Phi_{\tilde{z}}, A_{\tilde{w}}, -A_z, -A_w) = (A, \tilde{U} - U, -V, -U)$ in [34] after flipping $u \to -u$, and with the discussion around Eq. (57) in [4] by identifying $(\Phi_{\tilde{z}}^{(1)}, A_{\tilde{w}}^{(1)}, A_z^{(1)}, A_w^{(1)})$ with $(-B, -P, -H, -Q)$ in [4].

We note that $U(1)$ part of gauge groups must be always taken into account even when all gauge fields are traceless in some gauge as in (4.8) because gauge transformations could always give rise to trace parts. For example, explicit calculation of the gauge transformation (4.7) is

$$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}^{-1} \star \begin{pmatrix} a & b \\ c & d \end{pmatrix} \star \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = \begin{pmatrix} a + b \star h & 0 \\ c + d \star h - h \star a - h \star b \star h & d - h \star b \end{pmatrix}.$$

The trace increases by $[b, h]_\star$ after the transformation. The $U(1)$ part actually plays important roles in NC gauge theories and gives rise to new physical objects [1, 2, 3].

4.2 Reduction to NC mKdV equation

Here let us consider a symmetry reduction H_{+0}: $X = \partial_w - \partial_{\tilde{w}}, \ Y = \partial_{\tilde{z}}$ where the gauge group is $GL(2, \mathbb{R})$.

Now let us take further reduction condition on gauge fields:

$$\Phi_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \ A_z = \begin{pmatrix} c & b \\ 0 & d \end{pmatrix}, \ A_w = \begin{pmatrix} v & -1 \\ 0 & -v \end{pmatrix}. \tag{4.9}$$

The first equation of Eq. (4.1) is automatically satisfied. Eq. (4.1) becomes

$$\begin{pmatrix} v' + a + b & 0 \\ -a' + d - c + \{a, v\}_\star & -v' - a - b \end{pmatrix} = 0, \tag{4.10}$$

hence, $b = -v' - a, \ c - d = -a' + a \star v + v \star a$. The second equation of Eq. (4.1) becomes

$$\begin{pmatrix} c' - \dot{v} + [v, c]_\star & b' + c - d + \{b, v\}_\star \\ 0 & d' + \dot{v} - [v, d]_\star \end{pmatrix} = 0. \tag{4.11}$$

[8] This form is determined by the condition that the trace elements of $A_w^{(1)}$ must vanish.

From the 1-2 component of Eq. (4.11), we get

$$a = -\frac{1}{2}v' - \frac{1}{2}v \star v, \tag{4.12}$$

and hence

$$b = -\frac{1}{2}v' + \frac{1}{2}v \star v, \;\; c - d = \frac{1}{2}v'' - v \star v \star v. \tag{4.13}$$

From the trace of Eq. (4.11), we get $c + d = -(1/2)[v, v']_\star$. Therefore

$$c = \frac{1}{4}v'' - \frac{1}{2}v \star v \star v - \frac{1}{4}[v, v']_\star, \;\; d = -\frac{1}{4}v'' + \frac{1}{2}v \star v \star v - \frac{1}{4}[v, v']_\star. \tag{4.14}$$

Finally we get NC mKdV eq from the diagonal elements of Eq. (4.11):

$$\dot{v} = \frac{1}{4}v''' - \frac{3}{4}\left\{v \star v, v'\right\}_\star, \tag{4.15}$$

which is connected with NC KdV equation via NC Miura map: $u = v' - v^2$ [49]:

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u \star u' + u' \star u). \tag{4.16}$$

We comment that this reduction is gauge equivalent to that of NC KdV equation under the gauge transformation (4.7) with[9]

$$h_{(2)} = -\frac{1}{2}\left(v + \partial_x^{-1}(v \star v)\right), \quad (u =)\ 2q' = v' - v^2. \tag{4.17}$$

The transformed gauge fields are calculated as

$$\begin{aligned} &\Phi_{\tilde{z}}^{(2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; A_{\tilde{w}}^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ -v' - v \star v & 0 \end{pmatrix}, \; A_w^{(2)} = \frac{1}{4}\begin{pmatrix} v & -1 \\ 0 & -v \end{pmatrix}, \\ &A_z^{(2)} = \frac{1}{4}\begin{pmatrix} v'' - 2v \star v \star v - [v, v']_\star & -2v' + 2v \star v \\ 0 & -v'' + 2v \star v \star v - [v, v']_\star \end{pmatrix}. \end{aligned} \tag{4.18}$$

This is just the present reduction to NC mKdV which has infinite conserved quantities [49].

[9]The explicit form of $h_{(2)}$ and the relationship between q and v (just the NC Miura map) are determined by the condition that $A_w^{(2)}$ must coincide with A_w in (4.9).

4.3 Reduction to NC NLS equation

As we commented in Sec. 3.2, by the identification $\partial_w = \partial_{\tilde{w}}$, the reduction to NC Zakharov system coincides with that [28, 34] to NC NLS equation, that is, the following reduction conditions

$$\Phi_{\tilde{z}} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A_{\tilde{w}} = 0,\ A_w = \begin{pmatrix} 0 & \psi \\ \varepsilon\bar{\psi} & 0 \end{pmatrix},\ A_z = i\varepsilon\begin{pmatrix} -\psi\star\bar{\psi} & -\varepsilon\psi' \\ \bar{\psi}' & \bar{\psi}\star\psi \end{pmatrix}, \quad (4.19)$$

gives rise to NC NLS equation

$$i\dot{\psi} = \psi'' - 2\varepsilon\psi\star\bar{\psi}\star\psi \qquad (4.20)$$

where $\varepsilon = \pm 1$. The NC NLS equation (4.20) possesses infinite conserved quantities [53].

The KdV and NLS equations and the hierarchies possess the twistor descriptions [43, 54]. NC extension of them are very interesting and the details will be reported later.

4.4 Reduction to NC Boussinesq equation

Here we consider a symmetry reduction H_{SD}: $X = \partial_{\tilde{w}}$, $Y = \partial_{\tilde{z}}$ where the gauge group is $GL(3,\mathbb{R})$. The reduced ASDYM is the same as Eq. (4.2): Now let us take further reduction condition on gauge fields:

$$\Phi_{\tilde{z}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \Phi_{\tilde{w}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ A_z = \begin{pmatrix} a & 0 & -1 \\ d & b & 0 \\ f & e & c \end{pmatrix},\ A_w = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ v & u & 0 \end{pmatrix},$$

where all matrices are traceless. Eq. (4.2) gives rise to the following equations:

$$\begin{aligned} &a-b+u=0,\ b-c=0,\ a'-d+v=0,\ b'-e+d=0,\ c'-v+e=0,\ d'-f=0, \\ &f'-\dot{v}+v\star a+u\star d-c\star v=0,\ e'-\dot{u}+u\star b+f-c\star u=0. \end{aligned} \qquad (4.21)$$

Together with the traceless condition $a+b+c=0$, we get

$$a = -\frac{2}{3}u,\ b=c=\frac{1}{3}u,\ d=-\frac{2}{3}u'+v,\ e=-\frac{1}{3}u'+v,\ f=-\frac{2}{3}u''+v'. \qquad (4.22)$$

The remaining equations are

$$\dot{v} = -\frac{2}{3}u''' + v'' - \frac{2}{3}u\star u' + \frac{2}{3}[u,v]_\star,\quad \dot{u} = -u''+2v'. \qquad (4.23)$$

Eliminating v, we get NC Boussinesq equation [42]:

$$\ddot{u} + \frac{1}{3}u'''' + \frac{2}{3}(u\star u)'' - \frac{2}{3}([u,\partial_x^{-1}\dot{u}]_\star)' = 0. \qquad (4.24)$$

This coincides with that derived from NC Gelfand-Dickey hierarchy [55] and therefore possesses infinite conserved quantities [56] and exact multi-soliton solutions [51].

The potential form (NC potential Boussinesq equation) is obtained by setting $u = q'$:

$$\ddot{q} + \frac{1}{3}q'''' + \frac{2}{3}\{q', q''\}_\star + \frac{2}{3}[\dot{q}, q']_\star = 0. \tag{4.25}$$

In similar way, we can embed NC version of Drinfeld-Sokolov reductions [57] into a framework of NC ASDYM equation or NC twistor theory as in commutative case [8, 58, 59]. The detailed discussion will be reported later.

4.5 Reduction to NC N-wave equation

Here we consider a symmetry reduction H_{SD}: $X = \partial_{\tilde{w}},\ Y = \partial_{\tilde{z}}$ where the gauge group is $GL(N, \mathbb{R})$. If we take

$$\Phi_{\tilde{w}} = \mathrm{diag}(a_1, \cdots, a_N),\ \Phi_{\tilde{z}} = \mathrm{diag}(b_1, \cdots, b_N),\ (A_w)_{ij} = \omega_{ij},\ (A_z)_{ij} = \lambda_{ij}\omega_{ij}, \tag{4.26}$$

where $a_i, b_i, (i = 1, \cdots N)$ are constants and $\lambda_{ij} := (b_i - b_j)/(a_i - a_j) = \lambda_{ji},\ (i \neq j),\ \lambda_{ii} = 0$, and $\omega_{ij} = -\omega_{ji}$, we get NC N-wave equation [34]:

$$\dot{\omega}_{ij} = \lambda_{ij}\omega'_{ij} - \sum_{k=1}^{N}(\lambda_{ik} - \lambda_{kj})\omega_{ik} \star \omega_{kj}. \tag{4.27}$$

This reduction has the symmetry of reflection $\rho : w \mapsto w, z \mapsto z, \tilde{w} \mapsto -\tilde{w}, \tilde{z} \mapsto -\tilde{z}$. For the commutative reduction, see e.g. [8, 10].

4.6 Reduction to NC topological chiral model

Here we consider a symmetry reduction H_{SD}: $X = \partial_{\tilde{w}},\ Y = \partial_{\tilde{z}}$ where the gauge group is $GL(N, \mathbb{C})$. First we take a gauge in which $A_w = A_z = 0$. Then the linear systems for NC ASDYM equation becomes $L = \partial_x + \Phi_{\tilde{z}},\ M = \partial_t + \Phi_{\tilde{w}}$ at $\lambda = -1$, which implies that

$$\Phi_{\tilde{w}} = g^{-1} \star \partial_t g, \quad \Phi_{\tilde{z}} = g^{-1} \star \partial_x g, \tag{4.28}$$

where $g(t, x)$ is a nonsingular matrix-valued function of x and t. Then NC ASDYM equation is equivalent to

$$\partial_x(g^{-1} \star \partial_t g) - \partial_t(g^{-1} \star \partial_x g) = 0, \tag{4.29}$$

which is a NC version of topological chiral model [8]. This equation can be expressed as $d(g^{-1} \star dg) = 0$.

4.7 Reduction to NC (affine) Toda field equations

Here we consider a symmetry reduction H_{++}: $X = \partial_w = 0,\ Y = \partial_{\tilde{w}} = 0$ where the gauge group is $GL(N, \mathbb{C})$. If we take

$$
A_z = \begin{pmatrix} a_1 & & & O \\ & a_2 & & \\ & & \ddots & \\ O & & & a_N \end{pmatrix},\ A_{\tilde{z}} = \begin{pmatrix} -\tilde{a}_1 & & & O \\ & -\tilde{a}_2 & & \\ & & \ddots & \\ O & & & -\tilde{a}_N \end{pmatrix},
$$

$$
\Phi_w = \begin{pmatrix} 0 & \phi_1 & & & O \\ & 0 & \phi_2 & & \\ & & 0 & \ddots & \\ & O & & \ddots & \phi_{N-1} \\ \epsilon\phi_N & & & & 0 \end{pmatrix},\ \Phi_{\tilde{w}} = \begin{pmatrix} 0 & & & & \epsilon\tilde{\phi}_N \\ \tilde{\phi}_1 & 0 & & O & \\ & \tilde{\phi}_2 & 0 & & \\ & & \ddots & \ddots & \\ O & & & \tilde{\phi}_{N-1} & 0 \end{pmatrix}, \tag{4.30}
$$

where ϵ is a real constant which values 0 or 1, we get NC Toda equation for $\epsilon = 0$ [60]:

$$
\begin{aligned}
&\partial_z \phi_i = \phi_i \star a_{i+1} - a_i \star \phi_i, \quad \partial_{\tilde{z}} \tilde{\phi}_i = \tilde{a}_{i+1} \star \tilde{\phi}_i - \tilde{\phi}_i \star \tilde{a}_i, \quad (i = 1, 2, \cdots, N-1) \\
&\partial_z \tilde{a}_1 + \partial_{\tilde{z}} a_1 + [a_1, \tilde{a}_1]_\star + \phi_1 \star \tilde{\phi}_1 = 0, \\
&\partial_z \tilde{a}_i + \partial_{\tilde{z}} a_i + [a_i, \tilde{a}_i]_\star + \phi_i \star \tilde{\phi}_i - \phi_{i-1} \star \tilde{\phi}_{i-1} = 0, \quad (i = 2, 3, \cdots, N-1) \\
&\partial_z \tilde{a}_N + \partial_{\tilde{z}} a_N + [a_N, \tilde{a}_N]_\star - \phi_{N-1} \star \tilde{\phi}_{N-1} = 0,
\end{aligned} \tag{4.31}
$$

and NC affine Toda equation for $\epsilon = 1$ [60]:

$$
\begin{aligned}
&\partial_z \phi_i = \phi_i \star a_{i+1} - a_i \star \phi_i, \quad \partial_{\tilde{z}} \tilde{\phi}_i = \tilde{a}_{i+1} \star \tilde{\phi}_i - \tilde{\phi}_i \star \tilde{a}_i, \quad (i = 1, 2, \cdots, N-1) \\
&\partial_z \phi_N = \phi_N \star a_1 - a_N \star \phi_N, \quad \partial_{\tilde{z}} \tilde{\phi}_N = \tilde{a}_1 \star \tilde{\phi}_N - \tilde{\phi}_N \star \tilde{a}_N, \\
&\partial_z \tilde{a}_1 + \partial_{\tilde{z}} a_1 + [a_1, \tilde{a}_1]_\star + \phi_1 \star \tilde{\phi}_1 - \phi_N \star \tilde{\phi}_N = 0, \\
&\partial_z \tilde{a}_i + \partial_{\tilde{z}} a_i + [a_i, \tilde{a}_i]_\star + \phi_i \star \tilde{\phi}_i - \phi_{i-1} \star \tilde{\phi}_{i-1} = 0. \quad (i = 2, 3, \cdots, N).
\end{aligned} \tag{4.32}
$$

The NC (affine) Toda field equations have exact soliton solutions [60]. We note that the NC (A)SD Chern-Simons equation coupled to an adjoint matter just coincides with Eq. (4.3). For $N = 2$, the NC Toda equation and the NC affine Toda equation include NC Liouville equation and NC sinh-Gordon equation, respectively [60]. In the commutative limit, these equations reduce to ordinary form of (affine) Toda equations:

$$
\partial_z \partial_{\tilde{z}} u_i + \sum_j K_{ij} e^{u_j} = 0, \tag{4.33}
$$

where $u_i = \log(\phi_i \tilde{\phi}_i)$ and the matrix K_{ij} is an (extended) Cartan matrix associated to $SU(N)$. (The explicit definition is seen in e.g. [8].)

This reduction possesses an additional symmetry an in commutative case [8]. Let us take the rotational transformation $w \to w' = e^{i\theta}w$. Then the one-form $\Phi_w dw$ is transformed to $\Phi'_w dw' = G_\theta^{-1}\Phi_w G_\theta dw'$ where $G_\theta = \mathrm{diag}(1, \alpha^{-1}, \cdots, \alpha^{-(N-1)})$ and $\alpha = e^{i\theta}$. Φ'_w is calculated as

$$\Phi'_w = \alpha^{-1}\begin{pmatrix} 0 & \phi_1 & & & O \\ & 0 & \phi_2 & & \\ & & 0 & \ddots & \\ & O & & \ddots & \phi_{N-1} \\ \alpha^N \epsilon\phi_N & & & & 0 \end{pmatrix}. \tag{4.34}$$

Hence the one-form has the rotational symmetry generated by $Z = iw\partial_w - i\tilde{w}\partial_{\tilde{w}}$ for $\epsilon = 0$, and the discrete rotational symmetry where θ is integer multiples of $(2\pi/N)$ for $\epsilon = 1$. For the commutative reduction, see e.g. [8, 10].

4.8 Reduction to NC sine-Gordon equations

In this subsection, we review the reduction of NC Yang's equation to NC sine-Gordon equation where the gauge group is $GL(2,\mathbb{C})$. For the commutative discussion, see [61]. In such reductions, exponential functions are involved and NC extension of them has several possibilities. Naive definition of NC exponential functions is as follows

$$e_\star^\phi := \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\phi\star\cdots\star\phi}_{n \text{ times}}. \tag{4.35}$$

Let us start with the NC Yang's equation (2.14) and take the following ansatz:

$$J = e_\star^{i\sigma_1\tilde{w}} \star g(z,\tilde{z}) \star e_\star^{i\sigma_1 w}. \tag{4.36}$$

Then NC Yang's equation (2.14) reduces to

$$\partial_z(g^{-1}\star\partial_{\tilde{z}}g) - \left[\sigma_1,\ g^{-1}\star\sigma_1 g\right]_\star = 0. \tag{4.37}$$

This equation does not depend on w and $\tilde{w}$ and the present reduction is considered to belong to H_{++}. Pauli matrices are defined as usual:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.38}$$

From this equation, we can get two kind of NC sine-Gordon equations:

$$\begin{aligned} \partial_z\left(e_\star^{-iu/2}\star\partial_{\tilde{z}}e_\star^{iu/2}\right) &= 2i\sin_\star u \\ \partial_z\left(e_\star^{iu/2}\star\partial_{\tilde{z}}e_\star^{-iu/2}\right) &= -2i\sin_\star u \end{aligned} \tag{4.39}$$

for $g = \exp_\star\{(i/2)\sigma_3 u\}$ [31], and

$$\begin{aligned}
\partial_z\left(e_\star^{-iu/2}\star\partial_{\tilde z}e_\star^{iu/2}\right)+\partial_z\left(e_\star^{-iu/2}\star V\star e_\star^{iu/2}\right) &= 2i\sin_\star u\\
\partial_z\left(e_\star^{iu/2}\star\partial_{\tilde z}e_\star^{-iu/2}\right)+\partial_z\left(e_\star^{iu/2}\star V\star e_\star^{-iu/2}\right) &= -2i\sin_\star u,
\end{aligned} \tag{4.40}$$

for $g = \exp_\star\{(i/2)v\}\star\exp_\star\{(i/2)\sigma_3 u\}$ [32], where

$$\sin_\star u := \frac{e_\star^{iu}-e_\star^{-iu}}{2i},\ V := e_\star^{-iv/2}\star\partial_{\tilde z}e_\star^{iv/2}. \tag{4.41}$$

Both of them possess infinite conserved quantities [62, 31, 32] and the latter preserves, due to existence of two kind of fields, an integrable property of factorized S-matrix [32].

In the commutative limit, both of them reduce to sine-Gordon equation:

$$\partial_z\partial_{\tilde z}u = 4\sin u. \tag{4.42}$$

For (4.40), there is an additional decoupled equation $\partial_z\partial_{\tilde z}v = 0$.

4.9 Reduction to NC Liouville equation

In this subsection, we review the reduction of NC Yang's equation to NC Liouville equation by Cabrera-Carnero [33], where the gauge group is $GL(2,\mathbb{R})$. For the commutative discussion, see [61].

Let us start with the NC Yang's equation (2.14) and take the following ansatz:

$$J = e_\star^{\sigma-\tilde w}\star g(z,\tilde z)\star e_\star^{-\sigma+w}. \tag{4.43}$$

Then NC Yang's equation (2.14) reduces to

$$\partial_z(g^{-1}\star\partial_{\tilde z}g)-\left[\sigma_-,\ g^{-1}\star\sigma_+ g\right]_\star = 0, \tag{4.44}$$

where $\sigma_\pm := (1/2)(\sigma_1\pm i\sigma_2)$ and the present reduction is considered to belong to H_{++} for the same reason as the case for NC sine-Gordon equation. From this equation, we can get a NC Liouville equation:

$$\begin{aligned}
\partial_z\left(e_\star^{-\phi_+}\star\partial_{\tilde z}e_\star^{\phi_+}\right) &= e_\star^{-\phi_-}\star e_\star^{\phi_+}\\
\partial_z\left(e_\star^{-\phi_-}\star\partial_{\tilde z}e_\star^{\phi_-}\right) &= -e_\star^{-\phi_-}\star e_\star^{\phi_+},
\end{aligned} \tag{4.45}$$

for $g = \mathrm{diag}(\exp_\star\{\phi_+\},\exp_\star\{\phi_-\})$ where $\phi_+ = v+u,\ \phi_- = v-u$ [33]. This equation also has soliton solutions [63].

In the commutative limit, this reduce to Liouville equation:

$$\partial_z\partial_{\tilde z}u = e^{2u}, \tag{4.46}$$

together with a decoupled equation $\partial_z\partial_{\tilde z}v = 0$.

4.10 Reduction to NC Tzitzéica equation

Here let us discuss the reduction of NC Yang's equation to a NC Tzitzéica equation where the gauge group is $GL(3,\mathbb{R})$, which is new.

Let us start with the NC Yang's equation (2.14) and take the following ansatz:

$$J = e_\star^{-E_-\tilde{w}} \star g(z,\tilde{z}) \star e_\star^{E_+ w}, \tag{4.47}$$

where

$$E_- = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.48}$$

Then NC Yang's equation (2.14) reduces to

$$\partial_z(g^{-1} \star \partial_{\tilde{z}} g) - \left[E_-,\ g^{-1} \star E_+ g\right]_\star = 0. \tag{4.49}$$

As in Secs 4.8 and 4.9, the present reduction belongs to H_{++}, and leads to a NC version of Tzitzéica equation by taking $g = e_\star^\rho \star \mathrm{diag}(e_\star^\omega, e_\star^{-\omega}, 1)$:

$$\begin{aligned}
&\partial_z\left(e_\star^{-\omega} \star \partial_{\tilde{z}} e_\star^\omega\right) + \partial_z\left(e_\star^{-\omega} \star V \star e_\star^\omega\right) = e_\star^\omega - e_\star^{-2\omega} \\
&\partial_z\left(e_\star^{\omega} \star \partial_{\tilde{z}} e_\star^{-\omega}\right) + \partial_z\left(e_\star^{\omega} \star V \star e_\star^{-\omega}\right) = e_\star^{-2\omega} - e_\star^{\omega}, \\
&\partial_z V = 0,
\end{aligned} \tag{4.50}$$

where $V := e_\star^{-\rho} \star \partial_{\tilde{z}} e_\star^\rho$. The present reduction is a special case of reduction to NC affine Toda equations [33] and hence NC Tzitzéica equation possesses the same kind of good properties as NC affine Toda equation [33, 60].

In the commutative limit, these equations reduce to Tzitzéica equation:

$$\partial_z \partial_{\tilde{z}} \omega = e^\omega - e^{-2\omega}. \tag{4.51}$$

together with a decoupled equation $\partial_z \partial_{\tilde{z}} \rho = 0$. The present result is different from that by Dunajski [64].

4.11 Reduction to NC Ward's chiral and harmonic map equations

Here we discuss two real reductions of NC Yang's equation (2.14) by $H_{++} : X = \partial_z - \partial_{\tilde{w}}, Y = \partial_w + \partial_{\tilde{z}}$. By defining $u = z + \tilde{w}, v = w - \tilde{z}$, we get the following equation:

$$\partial_u(J^{-1} \star \partial_v J) + \partial_v(J^{-1} \star \partial_u J) = 0. \tag{4.52}$$

As in commutative case, we obtain two kind of reductions by imposing two different reality conditions for ultrahyperbolic signature.

- NC Ward's harmonic map equation

  By putting $\bar{w} = z, \bar{\tilde{z}} = -\tilde{z}$, so that $X = \bar{Y}$ and $u = \bar{v}$, the reduced equation is NC Ward's harmonic map equation: $J : \mathbb{C} \rightarrow G$:

$$\partial_u(J^{-1} \star \partial_{\bar{u}} J) + \partial_{\bar{u}}(J^{-1} \star \partial_u J) = 0, \tag{4.53}$$

  with the metric $ds^2 = 2(dzd\tilde{z} + d\bar{z}d\bar{\tilde{z}})$. This is a simple NC version of Ward's harmonic map equation [6].

- NC Ward's chiral equation in $(1+1)$-dimension

  By taking $w, z, \tilde{w}, \tilde{z}$ to be real, that is u, v to be also real, Eq. (4.52) is just the NC version of Ward's chiral equation [6].

For Euclidean signature, there is another choice of reality condition which leads to a NC version of harmonic map equation [8]. These equations have not yet been examined in detail. (See also [25, 27].)

5 Conclusion and Discussion

In the present paper, we proved that various NC integrable equations in both $(2+1)$ and $(1+1)$ dimensions are actually derived from NC ASDYM equations in the ultra-hyperbolic signature by reduction. Existence of these reductions guarantees that the lower-dimensional integrable equations actually have both corresponding physical situations, such as, reduced D0-D4 D-brane systems, and twistor descriptions of them. We can expect that analysis of exact NC soliton solutions could be applied to that of D-brane dynamics, and NC twistor theory would explain geometrical origin of NC integrable equations

There are mainly two steps to go further. One is to develop twistor descriptions of the present results. For example, twistor descriptions of KdV and NLS equations are given by Mason and Sparling [43, 54]. In their work, the relationship between Penrose-Ward transformation and inverse scattering method has been revealed. Inverse scattering method is one of the most traditional way to discuss their integrability in terms of action-angle variables. NC Penrose-Ward transformation for NC ASDYM equation has been

already developed by Kapustin, Kuznetsov and Orlov [14], and Brain and Hannabuss [16, 17, 18]. Hence NC extension of them could be possible and worth studying and will be reported later somewhere. Generalization of such twistor frameworks to $(2+1)$-dimension is developed by Strachan [48] and the NC extension is also interesting. In this case, noncommutativity could be introduced into space directions only and integrability could be defined as usual. In this sense, the twistor descriptions of them are expected to be easy to treat. Twistor interpretation of NC Drinfeld-Sokolov reductions and NC Bäcklund transformation presented in Sec. 2.4 should be developed to reveal a hidden symmetry of NC integrable hierarchies. It is also interesting to study a connection between the twistor description of the NC hierarchy and non-associative structure behind NC integrable systems developed by Dimakis and Müller-Hoissen [65].

Another direction is to reveal the corresponding physical situations in N=2 string theory and to apply exact analysis of NC integrable equations to them. The BPS equations in some D-brane configurations would just correspond to NC integrable equations and the soliton solutions correspond to lower-dimensional D-branes. In these situations, we can expect that similar applications to D-brane dynamics would be possible and successful. NC dressing and splitting methods would be also applicable to NC KdV equation and so on. NC solitons are sometimes so easy to treat that we could analyze energy densities of them and fluctuations around them and so on. This would be a hint to reveal the corresponding D-brane configurations which might be new BPS states such as [66].

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